

ON ROTATION OF COMPLEX STRUCTURES

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ABSTRACT. We put in a general framework the situations in which a Riemannian manifold admits a family of compatible complex structures, including hyperkähler metrics and the Spin-rotations of [3]. We determine the (polystable) holomorphic bundles which are rotatable, i.e., they remain holomorphic when we change a complex structure by a different one in the family.

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1. INTRODUCTION

Hyperkähler manifolds admit an S^2 family of complex structures, all of them integrable and compatible with the metric. This produces a collection of different complex manifolds, all of them naturally related, but very often with different algebro-geometric properties. For example, it is typical that some of the manifolds in the family are algebraic and others are not. Other properties, like the Hodge structures, also change in the family.

There are other situations in which a Riemannian manifold admits a family of compatible complex structures, like the $SU(4)$ -structures compatible with a $Spin(7)$ -structure on the 8-torus, studied in [3]. This consists of an S^6 family of complex structures, that is, a family of complex 4-tori all of them naturally related, and again with very different algebro-geometric properties. Indeed, in [3] there is an example of an abelian variety X with $\text{End}(X) = \mathbb{Q}[\sqrt{-d}] \times \mathbb{Q}[\sqrt{-d}]$, $d \in \mathbb{Z}_{>0}$ square-free, and another abelian variety X' in the same family with $\text{End}(X') = \mathbb{Q}[\sqrt{-d}, \sqrt{e}]$, $d, e \in \mathbb{Z}_{>0}$ square-free. Also, it is typical that some of the complex 4-tori in the family are algebraic whereas others are not.

In the present note, we aim to put both previous examples in a general framework. Moreover, we shall describe other instances of the same phenomena, like the case of the product of two K3 surfaces.

Let $E \rightarrow M$ be a (hermitian) complex vector bundle over a Kähler manifold (M, ω) . Then E admits a Hermitian-Yang-Mills connection (HYM connection, for short) if there

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is a hermitian connection A such that

$$\begin{cases} F_A \in \bigwedge^{1,1}(\text{End } E) \\ \Lambda F_A = \lambda \text{Id} \end{cases}$$

for a constant λ , where $\Lambda : \bigwedge^2 \rightarrow \bigwedge^0$ denotes contraction with ω . Decomposing $A = \partial_A + \bar{\partial}_A$ into $(1, 0)$ and $(0, 1)$ -components, we have that $\bar{\partial}_A$ is a holomorphic structure on E and moreover $(E, \bar{\partial}_A)$ is a polystable bundle with respect to ω (a direct sum of stable bundles all of the same slope). The reciprocal also holds: a polystable bundle with respect to ω admits a HYM connection. This is the content of the Hitchin-Kobayashi correspondence [6].

If M admits a family of complex structures compatible with the given metric, then $E \rightarrow M$ might be HYM with respect to all (or a subfamily) of the Kähler structures simultaneously. In the case of hyperkähler manifolds, such bundles are called hyperholomorphic and have been extensively studied by Verbitsky [8]. In the case of complex 4-dimensional tori with $\text{Spin}(7)$ -structures, such bundles have been described in [3], where they are called Spin-rotatable bundles.

A bundle E which is HYM with respect to different complex structures in one of these families is an interesting object, since it determines holomorphic bundles for different complex structures on the given (smooth) manifold M . Here, we shall call such bundles *rotatable*. In particular, the Chern classes of a rotatable bundle E are algebraic cycles on (M, J) for any of these complex structure J such that (M, J) is an algebraic complex manifold. This is an indirect route for constructing algebraic cycles. If this happens, we shall say that $c_j(E)$ are *rotatable algebraic cycles*.

Another instance in which rotations of complex structures have been used is [4]. Schlickewei has used this mechanism to determine Hodge classes in self-products of K3 surfaces which are rotatable algebraic cycles, thereby proving the Hodge conjecture in some cases.

We will describe the bundles which are HYM with respect to a family of complex structures compatible with a Riemannian structure (M, g) in the different situations of rotations of complex structures that we analyse.

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2. ROTATION OF COMPLEX STRUCTURES

Let M be a Riemannian manifold of real dimension $2n$, and let $H < \text{SO}(2n)$ be its holonomy group. Consider a second group G such that

$$H < G < \text{SO}(2n).$$

Here G has the role of a “ground” group, that is, we fix the G -structure of M . So if $G = \mathrm{SO}(2n)$, we are merely fixing the Riemannian structure of M .

A compatible complex structure is a reduction (parallel with respect to the Levi-Civita connection) to a group $U \cong \mathrm{U}(n)$ with $H < U < G$. This is equivalent to give a Kähler structure on M . We see this as follows: fix a base-point $p \in M$ and trivialize $T_p M = \mathbb{R}^{2n}$. A tensor T_p on $T_p M$ determines a parallel tensor T on M by doing parallel transport along curves, if and only if it is fixed by H . A complex structure on $T_p M$ is determined by $J_p : T_p M \rightarrow T_p M$ with $J_p^2 = -\mathrm{Id}$, which is equivalent to giving a subgroup $U < \mathrm{SO}(2n) = \mathrm{SO}(T_p M)$, where $U \cong \mathrm{U}(n)$ are the elements which fix J_p . Then J_p determines J with $\nabla J = 0$ (that is, an integrable complex structure) if and only if $H < U$.

We also consider the case of groups $U \cong \mathrm{SU}(n)$ with $H < U < G$ under the same terminology, although in this case M is endowed with a Kähler structure I *plus* a parallel form θ of type $(n, 0)$ with respect to I .

The set of compatible complex structures is thus

$$\mathcal{U} = \{U \mid H < U < G\}.$$

Changing a complex structure $U_1 \in \mathcal{U}$ to another one $U_2 \in \mathcal{U}$ will be called a *rotation of complex structures*.

We fix $U_0 \in \mathcal{U}$ and consider

$$N = \{g \in G \mid g H g^{-1} = H\}_o$$

and

$$C = \{g \in N \mid g U_0 g^{-1} = U_0\}_o,$$

where the subindex $_o$ means “connected component of the identity”. Clearly

$$H < C < N < G.$$

Conjugating U_0 via g produces another complex structure $U_g = g U_0 g^{-1}$. These complex structures are parametrized by

$$\mathcal{U}' = N/C.$$

Note that $\mathcal{U}' \subset \mathcal{U}$. In the situations of this paper, these sets are equal.

Now we will analyze different instances of rotations of complex structures.

3. HYPERKÄHLER ROTATIONS

3.1. K3 surfaces. K3 surfaces are Kähler surfaces with holonomy $H = \mathrm{SU}(2) = \mathrm{Sp}(1) < G = \mathrm{SO}(4)$. In particular K3 surfaces are hyperkähler.

The universal cover of $\mathrm{SO}(4)$ is $\widetilde{\mathrm{SO}(4)} = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, where $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ are two copies of $\mathrm{SU}(2) = \mathrm{Sp}(1)$. If we consider \mathbb{R}^4 as the space of quaternions \mathbb{H} , then $\mathrm{SU}(2)_L$ acts as the unit quaternions $\mathrm{Sp}(1) = S^3 \subset \mathbb{H}$ by multiplication on the left, and $\mathrm{SU}(2)_R$ acts by multiplication on the right.

The holonomy group of a K3 surface M is $H = \mathrm{SU}(2)_L < \mathrm{SO}(4)$. There are three complex structures I, J, K and $\{L = aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$ is the family

of all compatible complex structures on M . This family is a 2-sphere. Actually, the quaternions $i, j, k \in \text{Sp}(1) = \text{SU}(2)_R$, acting on the right on $\mathbb{H} = \mathbb{R}^4$, produce the tensors $I, J, K : TM \rightarrow TM$, by parallel transport.

Now fix the complex structure I . This is the same as to consider the subgroup

$$U_I = \text{U}(2) < \text{SO}(4)$$

of all elements of $\text{SO}(4)$ commuting with I . These are generated by $\text{SU}(2)_L$ and $S_I^1 = \{a \text{Id} + bI \mid a^2 + b^2 = 1\} \subset \text{SU}(2)_R$. So $U_I = \text{SU}(2)_L \cdot S_I^1$. We have then

$$\begin{aligned} N &= \text{SO}(4), \\ C &= U_I. \end{aligned}$$

The rotations of complex structures are given by

$$\mathcal{U}' = \text{SO}(4)/\text{U}(2).$$

Note that $\mathcal{U}' = \mathcal{U}$ in this case. Also

$$\mathcal{U}' = \text{SO}(4)/(\text{SU}(2)_L \cdot S_I^1) \cong \text{SU}(2)_R/S_I^1 \cong S^2.$$

The action of $\text{SU}(2)_R$ on \mathcal{U}' is by conjugation, and it moves all $L = aI + bJ + cK$ transitively.

Using the metric, we write $\text{End}(\mathbb{R}^4) \cong (\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^*$. The endomorphisms $aI + bJ + cK$, $(a, b, c) \in \mathbb{R}^3$, correspond to antisymmetric tensors, which are self-dual with respect to the Hodge $*$ -operator, that is, tensors in \bigwedge_+^2 . Otherwise said, $\text{SO}(4)$ acts on \bigwedge_+^2 , I corresponds to the Kähler form ω_I , the isotropy of ω_I is $U_I = \text{U}(2)$, and $\text{SO}(4)/\text{U}(2)$ is the orbit of ω_I in $\bigwedge_+^2 = \mathbb{R}^3$. This is the 2-sphere $S(\bigwedge_+^2)$, i.e.,

$$\text{SO}(4)/\text{U}(2) \cong S(\bigwedge_+^2) = S^2,$$

naturally. The action of $\text{SO}(4)/\text{SU}(2)_L = \text{SU}(2)_R/\pm \text{Id} = \text{SO}(3)$ is the standard action on this S^2 .

Suppose that $E \rightarrow M$ is a complex vector bundle with a connection which is HYM with respect to I . Then $F_A \in \bigwedge_I^{1,1}(\text{End } E)$ and $\Lambda_I F_A = \lambda \text{Id}$. We have a decomposition:

$$\bigwedge^2 = \bigwedge_+^2 \oplus \bigwedge_-^2 = \langle \omega_I, \omega_J, \omega_K \rangle \oplus \Delta_{I, \text{prim}}^{1,1}$$

(Here there is a slight abuse of notation: when referring to forms, \bigwedge^r means the bundle of r -forms on M ; when dealing with a vector space \mathbb{R}^n , \bigwedge^r is the r -th exterior power of $(\mathbb{R}^n)^*$. This will happen throughout.) From this it is clear that ω_J, ω_K span the space $\Delta_I^{2,0} = \text{Re}(\bigwedge_I^{2,0} \oplus \bigwedge_I^{0,2})$. Here $\Delta_I^{1,1} = \text{Re}(\bigwedge_I^{1,1})$ and $\Delta_{I, \text{prim}}^{1,1}$ is the space of primitive $(1,1)$ -forms (those orthogonal to ω_I).

There are two options:

- If $\lambda = 0$, then $F_A \in \bigwedge_{I, \text{prim}}^{1,1}(\text{End } E)$, so $F_A \in \bigwedge_{L, \text{prim}}^{1,1}(\text{End } E)$ for any $L \in \mathcal{U}$. Then E is HYM with respect to all $L \in \mathcal{U}$. Such bundle E is called hyperholomorphic in the terminology of [8]. Note that such bundle is rotatable with respect to all complex structures in $\mathcal{U} = S^2$.

- If $\lambda \neq 0$, then F_A is of type $(1, 1)$ only with respect to $\pm I$, and hence E is not rotatable.

3.2. Hyperkähler manifolds. The previous case generalizes to higher dimensions. Let M be a hyperkähler manifold of dimension $4n$. This means that M has a Riemannian metric whose holonomy is $H = \mathrm{Sp}(n) < \mathrm{SO}(4n)$. The group $\mathrm{Sp}(n)$ is the group of endomorphisms of $\mathbb{R}^{4n} = \mathbb{H}^n$ which commute with the quaternionic structure of \mathbb{H}^n as an \mathbb{H}_R -vector space (that is, \mathbb{H} acts on \mathbb{H}^n by multiplication on the right).

Therefore the elements of $\mathrm{Sp}(1) = S^3 \subset \mathbb{H}_R$, that is the quaternions of the form $ai + bj + ck$, $a^2 + b^2 + c^2 = 1$, produce endomorphisms $L = aI + bJ + cK$ on the tangent space TM which commute with the action of $H = \mathrm{Sp}(n)_L$, hence they are parallel with respect to the Levi-Civita connection. This gives an S^2 -family of complex structures on M compatible with the Riemannian metric.

Fix a complex structure I , given by some $U_I = \mathrm{U}(2n)$ with $\mathrm{Sp}(n) < U_I < \mathrm{SO}(4n)$. This subgroup is the isotropy of I , which is $U_I = \mathrm{Sp}(n) \cdot S_I^1$, where $S_I^1 = \{a \mathrm{Id} + bI \mid a^2 + b^2 = 1\}$. We have

$$\begin{aligned} N &= \mathrm{Sp}(n) \cdot \mathrm{Sp}(1), \\ C &= U_I = \mathrm{Sp}(n) \cdot S_I^1. \end{aligned}$$

Hence

$$\mathcal{U}' = N/C = \frac{\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)}{\mathrm{Sp}(n) \cdot S_I^1} \cong \mathrm{Sp}(1)/S_I^1 \cong S^2.$$

The following result gives us the decomposition of the space of 2-forms \bigwedge^2 under $\mathrm{Sp}(n)$. Consider the quaternionic space $V = \mathbb{R}^{4n} = \mathbb{H}^n$, with action of \mathbb{H} on the right. The space $W = \bigwedge^2 V$ consists of bilinear antisymmetric maps $\varphi : V \times V \rightarrow \mathbb{R}$. Let $W_{\mathbb{H}}$ be the subset of those bilinear maps such that $\varphi(xI, yI) = \varphi(xJ, yJ) = \varphi(xK, yK) = \varphi(x, y)$, for all $x, y \in V$; let W_I be the subset of those bilinear maps satisfying $\varphi(xI, yI) = -\varphi(xJ, yJ) = -\varphi(xK, yK) = \varphi(x, y)$, for all $x, y \in V$; define W_J and W_K similarly. Finally note that $\omega_I \in W_I$ produces an (orthogonal) decomposition $W_I = \langle \omega_I \rangle \oplus W_{I, \mathrm{prim}}$. Then

Lemma 1. *We have the following*

$$(1) \quad \bigwedge^2 = \langle \omega_I, \omega_J, \omega_K \rangle \oplus W_{\mathbb{H}} \oplus W_{I, \mathrm{prim}} \oplus W_{J, \mathrm{prim}} \oplus W_{K, \mathrm{prim}}.$$

With respect to the complex structure I ,

$$\begin{aligned} \Delta_{I, \mathrm{prim}}^{1,1} &= W_{\mathbb{H}} \oplus W_{I, \mathrm{prim}}, \\ \Delta_I^{2,0} &= \langle \omega_J, \omega_K \rangle \oplus W_{J, \mathrm{prim}} \oplus W_{K, \mathrm{prim}}. \end{aligned}$$

and analogously for the other complex structures.

Proof. We have to see that $W = W_{\mathbb{H}} \oplus W_I \oplus W_J \oplus W_K$. First, note that $W = \bigwedge^2 V$ has dimension $\dim W = 8n^2 - 2n$. Secondly, note that $W_{\mathbb{H}}, W_I, W_J, W_K$ are complementary subspaces, so their sum is a direct sum.

We introduce the following notation: for a quaternion $q = a + bi + cj + dk \in \mathbb{H}$, let $a = \operatorname{Re}(q)$, $b = \operatorname{Im}(q)$, $c = \operatorname{Jm}(q)$, $d = \operatorname{Km}(q)$. Take $A \in M_{n \times n}(\mathbb{H})$, and $\psi_A(x, y) = x^T A \bar{y}$. Then for A a real antisymmetric matrix, $\operatorname{Re}(\psi_A) \in W_{\mathbb{H}}$, and for A real symmetric, $\operatorname{Im}(\psi_A), \operatorname{Jm}(\psi_A), \operatorname{Km}(\psi_A) \in W_{\mathbb{H}}$. This implies that $\dim W_{\mathbb{H}} \geq 2n^2 + n$.

On the other hand, for A real antisymmetric, $\operatorname{Im}(\psi_{Ai}) \in W_I$, and for A real symmetric, $\operatorname{Re}(\psi_{Ai}), \operatorname{Jm}(\psi_{Ai}), \operatorname{Km}(\psi_{Ai}) \in W_I$. Hence $\dim W_I \geq 2n^2 - n$.

Analogously, $\dim W_J \geq 2n^2 - n$ and $\dim W_K \geq 2n^2 - n$. So $W_{\mathbb{H}} \oplus W_I \oplus W_J \oplus W_K$ has dimension at least $2n^2 + n + 3(2n^2 - n) = 8n^2 - 2n = \dim W$. This proves that $W = W_{\mathbb{H}} \oplus W_I \oplus W_J \oplus W_K$, and $\dim W_{\mathbb{H}} = 2n^2 + n$, $\dim W_I = \dim W_J = \dim W_K = 2n^2 - n$. \square

Note that a hyperkähler manifold (M, I) is holomorphically symplectic with symplectic form $\Omega_I = \omega_J + \sqrt{-1}\omega_K \in \bigwedge_I^{2,0}$, and $\omega_J, \omega_K \in \Delta_I^{2,0}$.

The action of $\operatorname{Sp}(1)$ on the set of complex structures of V acts on the decomposition (1) by rotating the first space and the last three summands. In particular,

$$\mathcal{U}' \cong S(\langle \omega_I, \omega_J, \omega_K \rangle) = S^2.$$

The main consequence of Lemma 1 is that

$$(2) \quad \Delta_L^{1,1} \cap \Delta_{L'}^{1,1} = W_{\mathbb{H}},$$

if $L, L' \in \mathcal{U}'$ and $L' \neq \pm L$.

If $E \rightarrow M$ is a complex vector bundle with a connection A which is HYM with respect to I , then $F_A \in \bigwedge_I^{1,1}(\operatorname{End} E)$ and $\Lambda_I F_A = \lambda \operatorname{Id}$. By (2), the connection A is HYM with respect to some $L \neq \pm I$ if and only if

$$F_A \in W_{\mathbb{H}}(\operatorname{End} E).$$

In the terminology of [8], such bundles are called hyperholomorphic. We have thus the following definition.

Definition 2. Let $E \rightarrow M$ be a complex vector bundle, and let A be a connection which is HYM with respect to I . We say that A is *hyperholomorphic* if $F_A \in W_{\mathbb{H}}(\operatorname{End} E)$.

Therefore, E is a rotatable bundle if and only if it is hyperholomorphic. In this case E is HYM with respect to all $L \in \mathcal{U}'$. Note that, in particular, it should be $\lambda = 0$.

We have a cohomological characterization of hyperholomorphic bundles as follows.

Proposition 3 ([7, Theorem 3.1]). *Let M be a compact hyperkähler manifold. and E is a vector bundle HYM with respect to I . Then E is hyperholomorphic if and only if $c_1(E), c_2(E)$ are Hodge classes with respect to J and K .*

Recall that a Hodge class with respect to some complex structure L is a class in $H_L^{p,p}(M)$, $p \geq 0$.

We have an alternative characterization of hyperholomorphic bundles in terms of calibrations of the Chern classes.

Theorem 4. *Let M be a compact hyperkähler manifold, and let E be a vector bundle HYM with respect to I with $\deg_I(E) = 0$. Then*

$$c_2(E) \cup [\omega_L]^{n-2} \leq c_2(E) \cup [\omega_I]^{n-2}$$

for $L \in \mathcal{U}'$ and E is HYM with respect to L if and only if there is equality.

Proof. We have the following

$$\alpha \wedge \alpha \wedge \frac{1}{(n-2)!} \omega_I^{n-2} = \begin{cases} \|\alpha\|^2 \text{vol}, & \alpha \in \Delta_I^{2,0} \\ -\|\alpha\|^2 \text{vol}, & \alpha \in \Delta_{I, \text{prim}}^{1,1} \\ (n-1)\|\alpha\|^2 \text{vol}, & \alpha \in \langle \omega_I \rangle \end{cases}$$

Therefore

$$\begin{aligned} c_2(E) \cup \frac{1}{(n-2)!} [\omega_I]^{n-2} &= \frac{1}{8\pi^2} \int_M \text{Tr}(F_A \wedge F_A) \wedge \frac{1}{(n-2)!} \omega_I^{n-2} \\ &= \frac{1}{8\pi^2} (\|F_A^{1,1,\text{prim}}\|^2 - \|F_A^{2,0}\|^2 - (n-1)\|\Lambda_I F_A\|^2) \\ &= \frac{1}{8\pi^2} (\|F_A\|^2 - 3\|F_A^{2,0}\|^2 - n\|\Lambda_I F_A\|^2), \end{aligned}$$

using that $\langle B, C \rangle = -\text{Tr}(BC)$ is the Killing metric in $\mathfrak{u}(r)$.

Therefore $c_2(E) \cup [\omega_L]^{n-2}$, $L \in \mathcal{U}$, achieves its maximum if $F_A^{2,0} = 0$ (w.r.t. L) and $\Lambda_L F_A = 0$. In this case A is HYM with respect to L . \square

Theorem 4 also appears as Claim 3.21 in [8] with a different proof.

4. COMPLEX TORI

4.1. Spin-rotation of complex 4-tori. Let $M = \mathbb{R}^8/\Lambda$ be a real 8-torus, where \mathbb{R}^8 is endowed with the standard Riemannian (flat) metric. Then the holonomy is trivial, $H = \{1\} < \text{SO}(8)$. We give M the $\text{Spin}(7)$ -structure given by the standard 4-form

$$\begin{aligned} \Omega &= dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\ &\quad - dx_{2358} - dx_{2367} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678} \end{aligned}$$

By definition, $G = \text{Spin}(7) < \text{SO}(8)$ is the isotropy subgroup of Ω .

We consider the $\text{SU}(4)$ -structures compatible with the $\text{Spin}(7)$ -structure, that is $U \cong \text{SU}(4)$ with $U < G$. An $\text{SU}(4)$ -structure on M is given by a complex structure I , compatible with the metric, and a $(4,0)$ -form $\theta \in \bigwedge^{4,0}$ with $|\theta| = 4$. The Kähler form is ω_I . The $\text{Spin}(7)$ -structure determined by U is given by the 4-form $\Omega_U = \frac{1}{2}\omega_I^2 + \text{Re}(\theta)$. We say that U is compatible with the given $\text{Spin}(7)$ -structure if $\Omega_U = \Omega$, or equivalently, if $U < G$. The space \mathcal{U} is the space of all such U .

Fix an $\text{SU}(4)$ -structure $U_0 = \text{SU}(4) < \text{Spin}(7)$ associated to (I, θ) . Then

$$\begin{aligned} N &= \text{Spin}(7), \\ C &= U_0 = \text{SU}(4). \end{aligned}$$

So the complex structures are parametrized by

$$\mathcal{U}' = N/C = \text{Spin}(7)/\text{SU}(4).$$

This space is a 6-sphere. It is described in [3, Lemma 1] as follows. The group $\text{Spin}(7)$ acts on the 2-forms, and the decomposition in irreducible summands is $\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{21}^2$, where Λ_7^2 is a 7-dimensional representation and it consists of those $\alpha \in \Lambda^2$ with $\Omega \wedge \alpha = 3 * \alpha$, and Λ_{21}^2 is a 21-dimensional representation and it consists of those $\alpha \in \Lambda^2$ with $\Omega \wedge \alpha = - * \alpha$. It is easy to see that $\omega_I \in \Lambda_7^2$. Then the action of $\text{Spin}(7)$ by conjugation on U_0 moves ω_I in Λ_7^2 transitively in the sphere $S(\Lambda_7^2)$ of elements of norm 2. That is,

$$\text{Spin}(7)/\text{SU}(4) \cong S(\Lambda_7^2) = S^6.$$

There is a map $\mathcal{L} : \Lambda^{2,0} \rightarrow \Lambda^{0,2}$ given by

$$\Lambda_I^{2,0} \cong (\Lambda_I^{2,0})^* \cong (\overline{\Lambda_I^{0,2}})^* \cong \Lambda_I^{0,2},$$

where the first map is the duality given by θ , the second map is conjugation, and the third map is given by the hermitian metric. This \mathcal{L} produces another map $\mathcal{L} : \Lambda_I^{0,2} \rightarrow \Lambda_I^{2,0}$, and by considering the real subspaces, a map $\mathcal{L} : \Delta_I^{2,0} \rightarrow \Delta_I^{2,0}$. It is easy to see that $\mathcal{L}^2 = \text{Id}$, so there is a decomposition $\Delta_I^{2,0} = \Delta_{I,+}^{2,0} \oplus \Delta_{I,-}^{2,0}$ into two 6-dimensional subspaces, according to the eigenvalues of \mathcal{L} . Then

$$\begin{aligned} \Lambda_7^2 &= \Delta_{I,+}^{2,0} \oplus \langle \omega_I \rangle \\ \Lambda_{21}^2 &= \Delta_{I,-}^{2,0} \oplus \Delta_{I,prim}^{1,1} \end{aligned}$$

as it is computed in [3, Proposition 2] (see also [1]). The conclusion is that given any $\gamma \in \Delta_{I,+}^{2,0}$, the form

$$\omega = 2 \frac{\omega_I + \gamma}{|\omega_I + \gamma|}$$

defines another $\text{SU}(4)$ -structure in \mathcal{U}' .

Let $E \rightarrow M$ be a hermitian complex vector bundle. Let A be a hermitian connection which is HYM with respect to I . Then $F_A \in \Lambda_I^{1,1}(\text{End } E)$ and $\Lambda_I F_A = \lambda \text{Id}$. We decompose $F_A = F_A^o + \frac{1}{r}(\text{Tr } F_A) \text{Id}$, where F_A^o is the trace-free part. We have that $c_1(E) = [\frac{\sqrt{-1}}{2\pi} \text{Tr } F_A]$ and

$$\beta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = \left[\frac{1}{8\pi^2} \text{Tr } (F_A^o \wedge F_A^o) \right].$$

Definition 5. A is a *spinstanton* (a $\text{Spin}(7)$ -instanton in the terminology of [2] or [3]) if $F_A^o \in \Lambda_{21}^2(\text{End } E)$.

There is a cohomological criterium for Spin-rotation as follows

Proposition 6. *Let $E \rightarrow M$ be a hermitian complex vector bundle. Let A be a connection which is HYM with respect to I . Then A is HYM with respect to $L \in \mathcal{U}'$ if and only if $c_1(E), c_2(E)$ are Hodge classes with respect to L .*

Proof. As A is HYM with respect to I , we have that $F_A^\circ \in \bigwedge_I^{1,1}(\text{End } E)$. In particular, $F_A^\circ \in \bigwedge_{21}^2(\text{End } E)$ and A is a spinstanton. By [3, Proposition 11], a spinstanton A is traceless HYM with respect to L (that is $F_A^\circ \in \bigwedge_L^{1,1}(\text{End } E)$) if and only if $\beta(E) \in H_L^{2,2}(M)$.

If $c_1(E) \in H_L^{1,1}(M)$, then $[\text{Tr } F_A]$ is of type $(1,1)$, so $\text{Tr } F_A = \beta + da$, for some $\beta \in \Omega^{1,1}(M)$, and a 1-form a . Changing the connection A to $A + a \text{Id}$, we have that $\text{Tr } F_A \in \bigwedge^{1,1}$, and hence $F_A \in \bigwedge^{1,1}(\text{End } E)$. So A is HYM with respect to L . \square

There is also a characterization of Spin-rotability in terms of calibrations, which is the analogue of Theorem 4 in this situation.

Theorem 7. *Let (M, I) be a complex 4-torus which is algebraic, and let $E \rightarrow M$ be a vector bundle which is HYM with respect to I . Assume that $c_1(E) = 0$. Then*

$$c_2(E) \cup [\omega_L]^2 \leq c_2(E) \cup [\omega_I]^2$$

with equality if and only if E is HYM with respect to L .

Proof. Let us recall the result of [3, Proposition 19]. Consider

$$(3) \quad k = \frac{\beta(E) \cup [\omega_I]^2}{[\omega_I]^4}.$$

Then

$$(4) \quad (\beta(E) - 3k[\omega_I]^2) \cup [\gamma]^2 \leq 0,$$

for any $\gamma \in \Delta_{I,+}^{2,0}$. There is equality if and only if E is traceless HYM with respect to L with $\omega_L = 2 \frac{\omega_I + \gamma}{|\omega_I + \gamma|}$.

Now let $\kappa^2 = |\omega_I + \gamma|^2 = 4 + |\gamma|^2$. So

$$\begin{aligned} \kappa^2 \beta(E) \cup [\omega_L]^2 &= 4\beta(E) \cup [\omega_I + \gamma]^2 \\ &= 4\beta(E) \cup ([\omega_I]^2 + 2[\omega_I] \cup [\gamma] + [\gamma]^2) \\ &\leq 4\beta(E) \cup [\omega_I]^2 + 12k[\omega_I]^2 \cup [\gamma]^2 \\ &= 4\beta(E) \cup [\omega_I]^2 + k|\gamma|^2[\omega_I]^4 \\ &= (4 + |\gamma|^2)\beta(E) \cup [\omega_I]^2 \\ &= \kappa^2 \beta(E) \cup [\omega_I]^2, \end{aligned}$$

using that $\beta(E) \cup [\omega_I] \cup [\gamma] = 0$ in the second line, (4) in the third line, $[\omega_I]^2 \cup [\gamma]^2 = 2|\gamma|^2 \frac{[\omega_I]^4}{4!}$ in the fourth line and the definition (3) of k in the fifth line. Hence

$$\beta(E) \cup [\omega_L]^2 \leq \beta(E) \cup [\omega_I]^2$$

with equality if and only if E is traceless HYM with respect to L . As $c_1(E) = 0$, $\beta(E) = c_2(E)$ and E is HYM with respect to L . \square

This result determines a sphere $S^r \subset S^6$, where $0 \leq r \leq 6$, (see [3, Proposition 17]), such that the bundle E is rotatable for the complex structures in this sphere. The

sphere S^r can be of different dimensions, depending on the bundle and manifold, as the examples in [3] show.

Moreover, there is an example in [3] of a complex torus (M, ω_I) and a rotatable bundle $E \rightarrow (M, \omega_I)$ for which there is a rotated structure L such that (M, ω_L) is, as a complex torus, of very different nature: for instance (M, ω_I) can be a decomposable complex abelian variety and (M, ω_L) be an indecomposable complex abelian variety.

4.2. Rotation of complex structures on tori. For a $2n$ -dimensional torus $M = \mathbb{R}^{2n}/\Lambda$ (with a flat Riemannian metric), we can consider the family of *all* complex structures compatible with the metric. This means that we take now $H = \{1\} < G = \mathrm{SO}(2n)$. Let $U_0 = \mathrm{U}(n) < G$ be one complex structure I . Then

$$\begin{aligned} N &= \mathrm{SO}(2n), \\ C &= \mathrm{U}(n). \end{aligned}$$

The family of complex structures on M is parametrized by

$$\mathcal{U}' = N/C = \mathrm{SO}(2n)/\mathrm{U}(n).$$

For a 4-torus, $\mathcal{U}' = \mathrm{SO}(4)/\mathrm{U}(2) \cong S^2$, and we recover the situation discussed previously for a hyperkähler rotation. This is due to the fact that a complex structure (a $\mathrm{U}(2)$ -structure) determines uniquely an $\mathrm{SU}(2)$ -structure. So the rotations of complex structures for a 4-torus are the same as the ones obtained by considering it as hyperkähler manifold. In [5], M. Toma has considered these rotations to construct new stable bundles on complex 2-tori.

For a $2n$ -torus with $2n > 4$, the situation is more complicated. For instance, for a 6-torus, the space

$$\mathcal{U}' = \mathrm{SO}(6)/\mathrm{U}(3) \cong \mathbb{C}P^3.$$

This means that the orbit of $\omega \in \bigwedge^2$ under $\mathrm{SO}(6)$ is diffeomorphic to $\mathbb{C}P^3$. However, it is difficult to describe it explicitly, since $\mathbb{C}P^3 \subset \bigwedge^2$ spans the whole of \bigwedge^2 , as this is an irreducible $\mathrm{SO}(6)$ -representation. Moreover, if $E \rightarrow M$ is a vector bundle endowed with an HYM connection A with respect to ω , then $F_A \in \bigwedge_I^{1,1}(\mathrm{End} E)$. For A to be HYM with respect to some other $L \in \mathcal{U}'$, we need to check that $F_A \in \bigwedge_L^{1,1}(\mathrm{End} E)$. This is a condition to be checked at every point $p \in M$, giving a functional equation. In the case of Spin-rotations for 8-tori, the real power of Theorem 7 is that it gives a *cohomological* condition for the functional equation $F_A \in \bigwedge_L^{1,1}(\mathrm{End} E)$ to hold everywhere.

If $E \rightarrow M$ is a bundle which is rotatable for the whole family $\mathrm{SO}(2n)/\mathrm{U}(n)$, that is, which is HYM for all complex structures in the family $\mathrm{SO}(2n)/\mathrm{U}(n)$, with $n > 2$, then A is flat, i.e. $F_A = 0$. This is shown in [9]. Note that however, it is possible to have a bundle $E \rightarrow M$ which is rotatable for a subfamily $\mathcal{F} \subset \mathrm{SO}(2n)/\mathrm{U}(n)$. For instance, take a Spin-rotatable bundle (there are examples in [3]) for a family $\mathcal{F} \subset \mathrm{Spin}(7)/\mathrm{SU}(4)$. Taking the image under the natural map $\mathrm{Spin}(7)/\mathrm{SU}(4) \xrightarrow{\iota} \mathrm{SO}(8)/\mathrm{U}(4)$, we get a bundle which is HYM for all complex structures in the family $\iota(\mathcal{F}) \subset \mathrm{SO}(8)/\mathrm{U}(4)$.

5. PRODUCT OF TWO K3 SURFACES

Let M, M' be two K3 surfaces. Then the holonomy of the manifold $X = M \times M'$ is $H = \mathrm{SU}(2) \times \mathrm{SU}(2) < \mathrm{SO}(4) \times \mathrm{SO}(4) < G = \mathrm{SO}(8)$. Fix complex structures I, I' on M, M' . This determines groups $U_I = \mathrm{U}(2) < \mathrm{SO}(4)$, $U_{I'} = \mathrm{U}(2) < \mathrm{SO}(4)$, and hence a subgroup $\mathrm{U}(2) \times \mathrm{U}(2) < \mathrm{SO}(4)$. We have a unique

$$U = U_I = \mathrm{U}(4) < \mathrm{SO}(8)$$

given by the complex structure $\mathcal{I} = I + I'$ on $X = M \times M'$. Then

$$\begin{aligned} N &= \mathrm{SO}(4) \times \mathrm{SO}(4), \\ C &= \mathrm{U}(2) \times \mathrm{U}(2). \end{aligned}$$

The quotient is

$$\mathcal{U}' = N/C = (\mathrm{SO}(4)/\mathrm{U}(2)) \times (\mathrm{SO}(4)/\mathrm{U}(2)) \cong S^2 \times S^2.$$

If I, J, K are the three complex structures of M and I', J', K' are the three complex structures of M' , then $\mathcal{L} = aI + bJ + cK + a'I' + b'J' + c'K'$, $(a, b, c), (a', b', c') \in S^2$, is a complex structure in the family \mathcal{U}' .

Write $\mathbb{R}^8 = V \oplus V'$, where $V, V' \cong \mathbb{R}^4$ correspond to the two factors M, M' . Then there is a decomposition into five irreducible components (under the group $N = \mathrm{SO}(4) \times \mathrm{SO}(4)$)

$$(5) \quad \bigwedge^2 = \langle \omega_I, \omega_J, \omega_K \rangle \oplus \Delta_{I, \text{prim}}^{1,1} V \oplus \langle \omega_{I'}, \omega_{J'}, \omega_{K'} \rangle \oplus \Delta_{I', \text{prim}}^{1,1} V' \oplus D,$$

where

$$D = \mathrm{Re} \left(\bigwedge_I^{1,0} V \otimes \bigwedge_{I'}^{1,0} V' \right) \oplus \mathrm{Re} \left(\bigwedge_I^{1,0} V \otimes \bigwedge_{I'}^{0,1} V' \right).$$

Note that, for the complex structure $\mathcal{I} = I + I'$, we have

$$\Delta_{\mathcal{I}}^{2,0} = \langle \omega_J, \omega_K \rangle \oplus \langle \omega_{J'}, \omega_{K'} \rangle \oplus \mathrm{Re} \left(\bigwedge_I^{1,0} V \otimes \bigwedge_{I'}^{1,0} V' \right).$$

Lemma 8. *Let $\alpha \in D$. For $(L, L') \in S^2 \times S^2$, we have*

$$-\int_X \alpha \wedge \alpha \wedge \omega_L \wedge \omega_{L'} \leq 4 \|\alpha\|^2,$$

and equality holds if and only if $\alpha \in \mathrm{Re} \left(\bigwedge_L^{1,0} V \otimes \bigwedge_{L'}^{0,1} V' \right)$.

Proof. Write $\alpha = \alpha_1 + \alpha_2 = \sum (a_i \wedge a'_i + \bar{a}_i \wedge \bar{a}'_i) + \sum (b_i \wedge \bar{b}'_i + \bar{b}_i \wedge b'_i)$, where $a_i, b_i \in \bigwedge_I^{1,0} V$ and $a'_i, b'_i \in \bigwedge_{I'}^{1,0} V'$. We have that $a_i \wedge \bar{a}_i \wedge \omega_L = -2\sqrt{-1}|a_i|^2 \mathrm{vol}_M$ and $a'_i \wedge \bar{a}'_i \wedge \omega_{L'} =$

$-2\sqrt{-1}|a'_i|^2 \text{vol}_{M'}$. Then

$$\begin{aligned} & \int_X \alpha \wedge \alpha \wedge \omega_L \wedge \omega_{L'} \\ &= 2 \sum \int_X a_i \wedge a'_i \wedge \bar{a}_j \wedge \bar{a}'_j \wedge \omega_L \wedge \omega_{L'} + 2 \sum \int_X b_i \wedge \bar{b}'_i \wedge \bar{b}_j \wedge b'_j \wedge \omega_L \wedge \omega_{L'} \\ &= -2 \sum \int_X a_i \wedge \bar{a}_j \wedge a'_i \wedge \bar{a}'_j \wedge \omega_L \wedge \omega_{L'} + 2 \sum \int_X b_i \wedge \bar{b}_j \wedge b'_j \wedge \bar{b}'_i \wedge \omega_L \wedge \omega_{L'} \\ &= 4 \int_X |\alpha_1|^2 \text{vol}_X - 4 \int_X |\alpha_2|^2 \text{vol}_X = 4||\alpha_1||^2 - 4||\alpha_2||^2. \end{aligned}$$

So $-\int_X \alpha \wedge \alpha \wedge \omega_L \wedge \omega_{L'} = 4||\alpha_2||^2 - 4||\alpha_1||^2 \leq 4||\alpha||^2$ and equality happens for $\alpha_1 = 0$. The result follows. \square

Suppose that $E \rightarrow X$ is a complex vector bundle with a connection A which is HYM with respect to $\mathcal{I} = I + I'$. Then $F_A \in \Lambda_{\mathcal{I}}^{1,1}(\text{End } E)$ and $\Lambda_{\mathcal{I}} F_A = \tilde{\lambda} \text{Id}$. Let

$$\begin{aligned} c_1(E) &= a + a' \in H^2(X) = H^2(M) \oplus H^2(M'), \\ c_2(E) &= b + b' + y \in H^4(X) = H^4(M) \oplus H^4(M') \oplus (H^2(M) \otimes H^2(M')). \end{aligned}$$

Let also $\lambda = a \cup [\omega_I]/[\omega_I]^2$, $\lambda' = a' \cup [\omega_{I'}]/[\omega_{I'}]^2$, so $\tilde{\lambda} = \frac{\lambda + \lambda'}{2}$. The following result tells us when E is rotatable.

Theorem 9. *E is rotatable only in the following cases:*

- $y = 0$, $\lambda = \lambda' = 0$. The rotations are given by the family $S^2 \times S^2$.
- $y = 0$, $\lambda = 0$, $\lambda' \neq 0$. The rotations are given by the family $S^2 \times \{\pm I'\}$.
- $y = 0$, $\lambda \neq 0$, $\lambda' = 0$. The rotations are given by the family $\{\pm I\} \times S^2$.
- $y \neq 0$, $\lambda = \lambda' = 0$. Then E is rotatable for those $\mathcal{L} = L + L' \in S^2 \times S^2$ such that

$$c_2(E) \cup [\omega_L] \cup [\omega_{L'}] = c_2(E) \cup [\omega_I] \cup [\omega_{I'}].$$

This family is either an S^2 embedded diagonally in $S^2 \times S^2$, or else E is not rotatable.

Proof. We decompose $F_A = F_1 + F_2 + F_3 + F_4 + F_5$ according to (5). Let $(L, L') \in S^2 \times S^2$ be another complex structure. We have to see if F_1, F_3 and F_5 are of type $(1, 1)$ with respect to $\mathcal{L} = L + L'$.

We start by noticing that $F_2 \wedge \omega_L = 0$ and $F_4 \wedge \omega_{L'} = 0$ for any (L, L') . Also $a \cup [\omega_L] = \frac{\sqrt{-1}}{2\pi} \int_M \text{Tr}(F_1) \wedge \omega_L = \frac{\sqrt{-1}}{2\pi} r \lambda [\omega_I] \cup [\omega_L]$, where $r = \text{rk}(E)$. Analogously, $a' \cup [\omega_{L'}] = \frac{\sqrt{-1}}{2\pi} r \lambda' [\omega_{I'}] \cup [\omega_{L'}]$. Then

$$\begin{aligned} c_2(E) \cup [\omega_L] \cup [\omega_{L'}] &= \frac{1}{8\pi^2} \int_X \text{Tr}(F \wedge F) \wedge \omega_L \wedge \omega_{L'} \\ &= \frac{1}{8\pi^2} \int_X \text{Tr}(F_5 \wedge F_5) \wedge \omega_L \wedge \omega_{L'} + \frac{2}{8\pi^2} \int_X \text{Tr}(F_1 \wedge F_3) \wedge \omega_L \wedge \omega_{L'} \\ &= -\frac{1}{8\pi^2} \int_X \langle F_5 \wedge F_5 \rangle \wedge \omega_L \wedge \omega_{L'} - \frac{1}{4\pi^2} r \lambda \lambda' ([\omega_I] \cup [\omega_L])([\omega_{I'}] \cup [\omega_{L'}]), \end{aligned}$$

using that $\langle A, B \rangle = -\text{Tr}(AB)$ is the Killing form on $\mathfrak{u}(r)$, the Lie algebra of $U(r)$.

Regarding the components F_1, F_3 , we have that $F_1 = \lambda \omega_I \text{Id}$, $F_3 = \lambda' \omega_{I'} \text{Id}$. If $\lambda, \lambda' \neq 0$, then F_1, F_3 are of type $(1, 1)$ with respect to $\mathcal{L} = L + L'$ only for the complex structures $\pm I \pm I'$. Therefore E is not rotatable.

If $\lambda \lambda' = 0$, then the formula above and Lemma 8 say that $F_5 \in \text{Re}(\bigwedge_L^{1,0} V \otimes \bigwedge_{L'}^{0,1} V')$ (End E) if and only if $c_2(E) \cup [\omega_L] \cup [\omega_{L'}]$ achieves its maximum. Considering

$$\begin{aligned} \Psi : S^2 \times S^2 &\longrightarrow \mathbb{R} \\ (\omega_L, \omega_{L'}) &\mapsto -\frac{1}{8\pi^2} \int_X \langle F_5 \wedge F_5 \rangle \wedge \omega_L \wedge \omega_{L'}, \end{aligned}$$

the maximum is achieved for $(\omega_I, \omega_{I'})$, by assumption. Note that Ψ is bilinear (when considered as a functional on $\mathbb{R}^3 \times \mathbb{R}^3$). It is easy to see that we can choose an orthonormal basis (that we shall call $\{I, J, K\}, \{I', J', K'\}$ again) in which Ψ has matrix

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

with $m_1 \geq m_2 \geq m_3$. If $m_1 > m_2$ then $\Psi(\omega_L, \omega_{L'}) = m_1$ only for $\pm(I + I')$. If $m_1 = m_2 > m_3$ then $\Psi(\omega_L, \omega_{L'}) = m_1$ for $L = aI + bJ$, $L' = aI' + bJ'$, for $a^2 + b^2 = 1$. Finally, if $m_1 = m_2 = m_3 > 0$ then $\Psi(\omega_L, \omega_{L'}) = m_1$ for $L = aI + bJ + cK$, $L' = aI' + bJ' + cK'$, for $a^2 + b^2 + c^2 = 1$.

Remark 10. Note that $\Psi \neq 0$ if and only if $m_1 \neq 0$. This is the same as to say $c_2(E) \cup [\omega_I] \cup [\omega_{I'}] \neq 0$, i.e., $y \cup [\omega_I] \cup [\omega_{I'}] \neq 0$. In particular, $y \neq 0 \iff y \cup [\omega_I] \cup [\omega_{I'}] \neq 0$.

Now, if either $\lambda = 0, \lambda' \neq 0$ or $\lambda \neq 0, \lambda' = 0$ then looking at the components F_1, F_3 , we have that E is rotatable only for $\mathcal{L} = L \pm I'$, $L \in S^2$, in the first case, or $\mathcal{L} = \pm I + L'$, $L' \in S^2$, in the second case. But then looking at F_5 , it must be $y = 0$ (this implying that $F_5 \equiv 0$).

If $\lambda = \lambda' = 0$, then $F_1, F_3 = 0$. So we only need to check that F_5 is of type $(1, 1)$ with respect to $\mathcal{L} = L + L'$. By the discussion above this happens exactly when

$$c_2(E) \cup [\omega_L] \cup [\omega_{L'}] = c_2(E) \cup [\omega_I] \cup [\omega_{I'}].$$

Choose the basis $\{I, J, K\}$ and $\{I', J', K'\}$ as above. Then E is rotatable for those $\mathcal{L} = L + L' = a(I + I') + b(J + J') + c(K + K')$ such that $y \cup [\omega_L] \cup [\omega_{L'}] = y \cup [\omega_I] \cup [\omega_{I'}]$. As $F_5 \in D(\text{End } E)$ is of type $(1, 1)$ with respect to $\mathcal{I} = I + I'$, we have that

$$F_5 \wedge F_5 \wedge (\omega_J + \sqrt{-1}\omega_K) \wedge (\omega_{J'} + \sqrt{-1}\omega_{K'}) = 0,$$

because $\omega_J + \sqrt{-1}\omega_K$ is of type $(2, 0)$. This means that $F_5 \wedge F_5 \wedge \omega_J \wedge \omega_{J'} = F_5 \wedge F_5 \wedge \omega_K \wedge \omega_{K'}$, implying that $m_2 = m_3$. This means that either E is not rotatable, or E is rotatable by an S^2 family embedded diagonally in $S^2 \times S^2$. \square

The rotability of E can be expressed in terms of the structure of holomorphic symplectic manifold. Recall that $\Omega_I = \omega_J + \sqrt{-1}\omega_K$, $\Omega_{I'} = \omega_{J'} + \sqrt{-1}\omega_{K'}$ and $\Omega_{\mathcal{I}} = \Omega_I + \Omega_{I'}$. We have the following

Corollary 11. *Let E be a hermitian vector bundle which is HYM with respect to $\mathcal{I} = I + I'$. Suppose that $\lambda = \lambda' = 0$. Then E is rotatable if and only if*

$$2c_2(E) \cup [\omega_{\mathcal{I}}]^2 = c_2(E) \cup [\Omega_{\mathcal{I}}] \cup [\overline{\Omega}_{\mathcal{I}}].$$

Proof. We have that

$$\begin{aligned} F_A \wedge F_A \wedge \Omega_{\mathcal{I}} \wedge \overline{\Omega}_{\mathcal{I}} &= 2\operatorname{Re}(F_5 \wedge F_5 \wedge (\omega_J + \sqrt{-1}\omega_K) \wedge (\omega_{J'} - \sqrt{-1}\omega_{K'})) \\ &= 2F_5 \wedge F_5 \wedge \omega_J \wedge \omega_{J'} + 2F_5 \wedge F_5 \wedge \omega_K \wedge \omega_{K'}. \end{aligned}$$

So

$$c_2(E) \cup [\Omega_{\mathcal{I}}] \cup [\overline{\Omega}_{\mathcal{I}}] = m_2 + m_3.$$

Then the condition of the statement is equivalent to $m_1 = m_2 = m_3$, which is equivalent to rotability, by Theorem 9. \square

Remark 12. Assume that a, a' are primitive forms. Then we have a Bogomolov type inequality: $c_2(E) \cup [\omega_{\mathcal{I}}]^2 \geq 0$, and this is equal to 0 if and only if $c_2(E) = b + b'$.

In [4], Schlickewei uses these rotations for the self-product of a K3 surface, $X = M \times M$, but considering only complex structures which are self-products of a complex structure on the K3 surface, that is, restricting consideration to the diagonal $\Delta \subset S^2 \times S^2$. Such X can be treated then as a hyperkähler manifold with the arguments of Section 3.

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